BIT 30 (1990), 332–346

NEWTON INTERPOLATION AT LEJA POINTS*

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Abstract.

The Newton form is a convenient representation for interpolation polynomials. Its sensitivity to perturbations depends on the distribution and ordering of the interpolation points. The present paper bounds the growth of the condition number of the Newton form when the interpolation points are Leja points for fairly general compact sets K in the complex plane. Because the Leja points are defined recursively, they are attractive to use with the Newton form. If K is an interval, then the Leja points are distributed roughly like Chebyshev points. Our investigation of the Newton form defined by interpolation at Leja points suggests an ordering scheme for arbitrary interpolation points.

AMS(MOS) Subject Classification: 65D05

Key words and phrases: polynomial interpolaton, Newton form, stability, Leja points, ordering of interpolation points.

1. Introduction.

Let C denote the complex plane, and identify C with the real plane R^2. Throughout this paper $x, y \in R$ and $z \in C$. If $z = x + iy$, then (x, y) and z denote the same point. Let $K \subset C$ be a compact set, whose complement $K_c := (C \cup \{\infty\})\backslash K$ is connected and possesses a Green's function $G(x, y)$ with a pole at infinity. The Green's function $G(x, y)$ is uniquely determined by the requirements i) $\Delta G(x, y) = 0$ in $K_c\backslash\{\infty\}$, ii) $G(x, y) = 0$ on ∂K, where ∂K denotes the boundary of K, and iii) $\frac{1}{2\pi}\int_{\partial K} \frac{\partial G}{\partial n}(x, y)ds = 1$, where $\frac{\partial}{\partial n}$ denotes the normal derivative directed into K_c; see [14, Chap. 4.1] for details. The nonnegative constant c defined by

$$\lim_{|z|\to\infty} |z| \exp(-G(x, y)) =: c, \qquad z = x + iy, \tag{1.1}$$

is called the *capacity* of K. The capacity depends on the scaling of K, i.e. if K has

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* Research supported in part by NSF under Grant DMS-8704196 and by U.S. Air Force Grant AFSOR-87-0102.

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Received May 1989. Revised August 1989.

capacity c, and α is a positive constant, then $\alpha K := \{\alpha z : z \in K\}$ has capacity αc.

EXAMPLE 1.1. Let $K := \{z : |z| \leq r\}$ for some constant $r > 0$. Then $G(x, y) = \ln|z/r|$, where $z = x + iy$, and therefore K has capacity r.

EXAMPLE 1.2. Let $K := [-a, a]$, $a > 0$. With $z = x + iy$, we obtain $G(x, y) = \ln|(z/a) + ((z/a)^2 - 1)^{1/2}|$, where the branch of the square root is chosen so that $|(z/a) + ((z/a)^2 - 1)^{1/2}| \geq 1$ for $z \in K_c$. The capacity of K is therefore $a/2$. For later reference we note in particular that the interval $[-2, 2]$ has capacity 1.

Leja (see [9] or [6], p. 178]) introduced a recursively defined sequence $\{z_j\}_{j=0}^{\infty}$ of points in K as follows. Let z_0 be an arbitrary but fixed point in K, and select points z_{j+1} so that

$$\prod_{k=0}^{j} |z_{j+1} - z_k| = \max_{z \in K} \prod_{k=0}^{j} |z - z_k|, \qquad z_{j+1} \in K, \quad j = 0.1, 2, \ldots. \tag{1.2a}$$

We will choose $z_0 \in K$ so that

$$|z_0| = \max_{z \in K} |z|. \tag{1.2b}$$

The points z_j are in general not uniquely determined by (1.2). We call any points z_j, $j = 0, 1, 2, \ldots$, that satisfy (1.2) *Leja points* for K. By the maximum principle, the Leja points lie on ∂K.

EXAMPLE 1.3. Let $K := \{z : |z| \leq 1\}$ and choose $z_0 = 1$. The next three Leja points z_1, z_2 and z_3 are defined as follows. The point z_1 is given by $z_1 := -1$, and we can select $z_2 := i$ or $z_2 := -i$. Then $z_3 := -z_2$. This illustrates the non-uniqueness.

Now let the non-negative integer k have the binary representation

$$k = \sum_{j=0}^{\infty} k_j 2^j, \qquad k_j \in \{0, 1\}, \tag{1.3a}$$

and introduce

$$z_k := \exp(i\pi \sum_{j=0}^{\infty} k_j 2^{-j}). \tag{1.3b}$$

It can be verified that the points z_k, $k \geq 0$, given by (1.3) are Leja points for K. Note that the points z_k are obtained by bit-reversal of the binary representation of k. Interpolation at the points defined by (1.3) has previously been discussed in [10], where it was noted that the arguments of the points (1.3b) form a Van der Corput sequence.

Table 1.1. *Arguments of the points* (1.3*b*).

k	0	1	2	3	4	5	6	7
$\arg(z_k)$	0	π	$\pi/2$	$3\pi/2$	$\pi/4$	$5\pi/4$	$3\pi/4$	$7\pi/4$

Section 2 reviews results of Leja [9] on the distribution of Leja points for compact point sets K. Let f be a funtion, analytic in an open set containing K. Let Π_n denote the set of polynomials of degree at most n, and let $p_n \in \Pi_n$ be the unique polynomial that interpolates f in a set of Leja points $\{z_j\}_{j=0}^n$ for K. Leja [9] shows that the polynomials p_n, $n \geq 0$, converge to f on K with an optimal geometric rate of convergence; see Section 2. Therefore Leja points are suitable interpolation points.

We represent the interpolation polynomials p_n by the *Newton form*

$$p_n(z) = [z_0]f + \sum_{j=1}^{n} ([z_0, z_1, \ldots, z_j]f) \prod_{k=0}^{j-1} (z - z_k), \tag{1.4a}$$

where

$$[z_j]f(z_j), \quad 0 \leq j \leq n, \tag{1.4b}$$

and

$$[z_j, z_{j+1}, \ldots, z_k]f := ([z_j, z_{j+1}, \ldots, z_{k-1}]f - [z_{j+1}, z_{j+2}, \ldots, z_k]f)(z_j - z_k)^{-1}, \quad 0 \leq j < k \leq n. \tag{1.4c}$$

The Newton form is an attractive representation for interpolation polynomials because it can be determined and evaluated rapidly, and, moreover, it is easy to determine p_{n+1} if p_n already is known; see, e.g., de Boor [3, Chap. 1] for details. In Section 2 we bound the growth of the condition number of the Newton form (1.4) when the z_j are Leja points. Our analysis shows that in order to achieve high accuracy, K should be scaled to have capacity 1.

The numerical determination of Leja points from the definition (1.2) is generally cumbersome. In computations we therefore replace the set K in (1.2) by a discrete point set $S_m \subset \partial K$ consisting of m distinct points, and use Leja points for S_m as interpolation points, i.e. we determine interpolation points $z_0, z_1, \ldots, z_n$, $n < m$, such that

$$|z_0| = \max_{z \in S_m} |z|, \quad z_0 \in S_m, \tag{1.5a}$$

$$\prod_{k=0}^{j} |z_{j+1} - z_k| = \max_{z \in S_m} \prod_{k=0}^{j} |z - z_k|, \quad z_{j+1} \in S_m, \quad 0 \leq j < n < m. \tag{1.5b}$$

For moderate values of m the determination of Leja points for S_m is a much simpler computational task than finding Leja points for general compact sets K. Note that since S_m in (1.5) is allowed to be an arbitrary set of m distinct points in C, we can use (1.5) to order m arbitrary interpolation points.

The advantage of ordering the interpolation points in the Newton form according to (1.5) are illustrated by Example 4.1 of Section 4. Other examples motivating an ordering of interpolation points can be found in [10] and [12], where interpolation in Fejér points and in extreme points of Chebyshev polynomials on $[-2, 2]$ are considered. Examples with polynomials in Newton form of very high degree are presented by Tal-Ezer [13].

Let f be a function analytic on K, and let S_m be a set of m distinct points on ∂K. In Section 3 we consider the approximation of f on K in the uniform norm by polynomials $p_n \in \Pi_n$ interpolating f at Leja points for S_m. The set S_m is typically chosen so that points in S_m are easy to determine numerically. For instance, if K is a rectangle, then we may choose S_m to be a set of equidistant points on ∂K. The interpolation polynomial p_n generally approximates f well on K, if the number of points in S_m is chosen sufficiently large compared with the degree n of p_n. Computed examples are presented in Section 4. These examples suggest that polynomial interpolation at Leja points for a set $S_m \subset \partial K$ may be an attractive method for computing polynomial approximants to f on K.

We conclude this section by briefly mentioning a least-squares method for determining polynomials that approximate analytic functions f on K in the uniform norm. Let $S_m = \{z_k\}_{k=0}^{m-1}$ consist of distinct points on ∂K, and let q_n be the unique polynomial in Π_n that satisfies $\|f - q_n\|_2 = \min_{p \in \Pi_n} \|f - p\|_2$, where $\|g\|_2 := (\sum_{k=0}^{m-1} |g(z_k)|^2)^{1/2}$. If the degree n of q_n is sufficiently small in relation to the number of points m in S_m, and if the points z_k are distributed fairly uniformly over all of ∂K, then q_n will approximate f well on K in the uniform norm. The case when K is an interval and S_m consists of m equidistant points has been considered by Björk [2], who found that n should not be selected much larger than $2m^{1/2}$. These results are also discussed by Dahlquist and Björck [4, p. 128]. The case when K is a simply connected compact set in C is considered in [11]. In this least-squares scheme one has to choose a well-conditioned polynomial basis. The operation count for solving the least-squares problem is in general $O(mn^2)$. For comparison, we note that the scheme of the present paper yields a fairly well-conditioned polynomial basis, and requires only $O(mn)$ arithmetic operations.

2. Newton interpolation at Leja points.

We first review some results on the distribution of Leja points. These results are used to bound the condition number of the Newton form. Throughout this section we assume that K satisfies the conditions of the first paragraph of Section 1.

LEMMA 2.1. *Let $\{z_j\}_{j=0}^{\infty}$ be a sequence of Leja points for K, i.e. the z_j satisfy* (1.2). *Then*

$$\prod_{j=0}^{n-1} |z_n - z_j|^{1/n} \geq c \tag{2.1}$$

and

$$\lim_{n\to\infty} \prod_{j=0}^{n-1} |z_n - z_j|^{1/n} = c, \tag{2.2}$$

where c is the capacity of K.

PROOF. Let $G(x, y)$ be the Green function for K_c introduced in Section 1, and let $H(x, y)$ be a conjugate harmonic function to $G(x, y)$. Then the maximum principle holds for

$$\phi(z) := \prod_{j=0}^{n-1} (z - z_j) \exp(-nG(x, y) - inH(x, y)), \qquad z = x + iy, \quad z \in K_c \cup \partial K.$$

By (1.1) we obtain $|\phi(\infty)| \geq c^n$, and therefore $\max_{z \in \partial K} |\phi(z)| \geq c^n$. This shows (2.1). Formula (2.2) is shown by Leja [9, Lemma 1]. ■

LEMMA 2.2. (*Leja* [9, *Theorem* 1]). *Let* $\{z_j\}_{j=0}^{\infty}$ *be a sequence of Leja points for* K, *and let* $z \in K_c$. *Then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln |z - z_j| = G(x, y) + \ln c, \qquad z = x + iy, \tag{2.3}$$

where c *is the capacity of* K, *and* G *is the Green's function for* K_c *introduced in Section* 1. *The limit is uniform for* z *belonging to any compact subset of* K_c.

The existence of the limit (2.3) can be used to establish convergence of polynomials interpolating analytic functions at Leja points. Introduce the norm

$$\|g\|_{\partial K} := \max_{z \in \partial K} |g(z)| \tag{2.4}$$

for functions analytic in $K \backslash \partial K$ and continuous on K. Define level curves in K_c,

$$L(\rho) := \{z = x + iy : G(x, y) = \rho\}, \qquad \rho > 0, \tag{2.5}$$

of the Green function G for K_c.

LEMMA 2.3. *Let* $\rho > 0$ *be the largest constant such that* f *is analytic and single-valued in the interior of the level curve* $L(\rho)$. *Let* $\{z_j\}_{j=0}^{\infty}$ *be a sequence of Leja points for* K, *and let* $p_n \in \Pi_n$ *be the unique polynomial that interpolates* f *at the points* $z_0, z_1, \ldots, z_n$. *Then*

$$\overline{\lim_{n \to \infty}} \|f - p_n\|_{\partial K}^{1/n} = e^{-\rho}. \tag{2.6}$$

Moreover, there is no sequence of polynomials $\{q_n\}_{n=0}^{\infty}$, $q_n \in \Pi_n$, *such that* $\overline{\lim}_{n \to \infty} \|f - q_n\|_{\partial K}^{1/n} < e^{-\rho}$.

PROOF. The proof follows by combining Lemma 2.2 with [14, Theorem 2, p. 154.] ■

Assume for the moment that ∂K consists of a finite number of mutually exterior Jordan curves. Then, combining Lemma 2.3 and [14, Theorem 5, p. 168] shows that the Leja points are uniformly distributed on ∂K with respect to the density function

$(2\pi)^{-1}\,\partial G/\partial n$. The Leja points share this property with the better known Fekete points for K. The latter are considered in [14, Theorem 6, pp. 170–171].

We are now in a position to bound the growth of the condition number of the Newton form when the interpolation points are Leja points for K. Related investigations for other polynomial bases have been carried out by Gautschi; see [7] and references therein. In order to specify the analyticity requirements for the interpolated functions, we have to introduce some notation. Let $A_{\rho,M}$ denote the set of functions f that are analytic on the closed set bounded by the level curve $L(\rho)$ and satisfy $\|f\|_{L(\rho)} \leq M$, where ρ and M are positive constants. Let $\{z_j\}_{j=0}^{\infty}$ be a sequence of Leja points for K, and introduce, for positive constants ρ and M, the sets

$$C_{\rho,M}^{n+1} := \{\mathbf{f} := (f(z_0), f(z_1), \ldots, f(z_n))^T : f \in A_{\rho,M}\},$$

$$\Pi_{n,\rho,M} := \{p \in \Pi_n : p(z_j) = f(z_j), \quad 0 \leq j \leq n, \quad f \in A_{\rho,M}\}.$$

Clearly, $C_{\rho,M}^{n+1} \subset C^{n+1}$ and $\Pi_{n,\rho,M} \subset \Pi_n$.

Any vector $\mathbf{g} = (g_0, g_1, \ldots, g_n)^T \in C^{n+1}$ may be regarded as the tabulation of some function g at the first $n+1$ Leja points z_j of K, i.e. we may think of $\mathbf{g}$ as being given by $g_j = g(z_j)$ for $0 \leq j \leq n$. This enables us to define differences $[z_j, z_{j+1}, \ldots, z_k]\mathbf{g}$ for $0 \leq j \leq k \leq n$, analogously with (1.4b)-(1.4c). We introduce an operator $T_0 : C^{n+1} \to C^{n+1}$ that maps a vector onto the vector of divided differences

$$T_0\mathbf{g} := ([z_0]\mathbf{g}, [z_0, z_1]\mathbf{g}, \ldots, [z_0, z_1, \ldots, z_n]\mathbf{g})^T, \qquad \mathbf{g} \in C^{n+1}.$$

We can now define the mappings used to bound the condition number of the Newton form. Let ρ and M be positive constants and introduce

$$T_1 : C_{\rho,M}^{n+1} \to C^{n+1}, \qquad T_1\mathbf{f} := T_0\mathbf{f}, \tag{2.7}$$

$$T_2 : C^{n+1} \to \Pi_n, \qquad (T_2\mathbf{d})(z) := d_0 + \textstyle\sum_{k=1}^{n} d_k \prod_{j=0}^{k-1} (z - z_j), \tag{2.8}$$

$$T : C_{\rho,M}^{n+1} \to \Pi_n, \qquad T := T_2 \circ T_1, \tag{2.9}$$

where $\mathbf{d} = (d_0, d_1, \ldots, d_n)^T$. Then $(T\mathbf{f})(z) = p_n(z)$, where $\mathbf{f} = (f(z_0), f(z_1), \ldots, f(z_n))^T$ for some function $f \in A_{\rho,M}$ and $p_n(z)$ is given by (1.4). Because T maps $C_{\rho,M}^{n+1}$ onto $\Pi_{n,\rho,M}$ in a one-to-one manner, we can define the inverse mapping $T^{-1} : \Pi_{n,\rho,M} \to C_{\rho,M}^{n+1}$. Equip the domains of T_1, T_2 and T, as well as the ranges of T_1 and T^{-1}, with the vector norm

$$\|\mathbf{v}\|_\infty := \max_{0 \leq j \leq n} |v_j|, \qquad \mathbf{v} = (v_0, v_1, \ldots, v_n)^T.$$

Let the domain of T^{-1} and the ranges of T_2 and T have the norm (2.4). Let $\|T_1\|$, $\|T_2\|$, $\|T\|$ and $\|T^{-1}\|$ denote the induced operator norms. The condition number of T is defined by

$$\operatorname{cond}(T) := \|T\| \, \|T^{-1}\|.$$

For many sequences of interpolation points the condition number cond(T) grows exponentially with the number of points. Example 4.1 below provides an illustration. The following theorem shows that for Leja points the condition number cond(T) grows slower than exponentially with the number of Leja points.

THEOREM 2.4. *Let K satisfy the conditions of the first paragraph of Section* 1. *Let $\{z_j\}_{j=0}^{\infty}$ be a sequence of Leja points for K. Then*

$$\lim_{n\to\infty} (\mathrm{cond}(T))^{1/n} = 1. \tag{2.10}$$

PROOF. We assume that K has capacity 1. The minor modifications of the proof required if the capacity of K is different from 1 are discussed below. We first bound $\|T_1\|$. Let ρ and M be arbitrary but fixed positive constants, and let $\mathbf{f} = (f_0, f_1, \ldots, f_n)^T \in C_{\rho,M}^{n+1}$. Then there is a function $f \in A_{\rho,M}$ such that $f_j = f(z_j)$ for $0 \le j \le n$, and therefore, the divided differences of $\mathbf{f}$ can be represented by

$$[z_0, z_1, \ldots, z_j]\,\mathbf{f} = \frac{1}{2\pi i}\int_{K(\rho)} \frac{f(\zeta)}{\prod_{k=0}^{j}(\zeta - z_k)}\,d\zeta, \qquad 0 \le j \le n, \tag{2.11}$$

see, e.g., [5, Chap. 3.6]. From (2.11) it follows that

$$|[z_0, z_1, \ldots, z_j]\mathbf{f}| \le \frac{M}{2\pi}\int_{L(\rho)} |d\zeta|\, \Big\|\prod_{k=0}^{j}(z - z_k)^{-1}\Big\|_{L(\rho)}, \qquad 0 \le j \le n,$$

and therefore,

$$\|T_1\| \le \left(\frac{M}{2\pi}\int_{L(\rho)} |d\zeta|\right) \max_{0\le j\le n} \Big\|\prod_{k=0}^{j}(z - z_k)^{-1}\Big\|_{L(\rho)}. \tag{2.12}$$

Because the set bounded by $L(\rho)$ has capacity larger than 1, we obtain by Lemma 2.2 that there is a constant γ_0, depending on ρ, but independent of j, such that

$$\Big\|\prod_{k=0}^{j}(z - z_k)^{-1}\Big\|_{L(\rho)} \le \gamma_0, \qquad j \ge 0. \tag{2.13}$$

Substitution of (2.13) into (2.12) shows that there is a constant γ_1, independent of n, such that

$$\|T_1\| \le \gamma_1. \tag{2.14}$$

We turn to a bound for $\|T_2\|^{1/n}$. By the definition (2.8) of T_2, it follows that

$$\|T_2\| = \max_{\|\mathbf{d}\|_\infty = 1} \|T_2\mathbf{d}\|_{\partial K} \le 1 + \sum_{j=1}^{n} \Big\|\prod_{k=0}^{j-1}(z - z_k)\Big\|_{\partial K},$$

and the definition (1.2) of Leja points for K yields

$$\|T_2\| \le 1 + \sum_{j=1}^{n}\prod_{k=0}^{j-1}|z_j - z_k|. \tag{2.15}$$

From (2.2) it follows that for any constant $\rho > 1$ there is a constant $\gamma_2 \geq 1$, depending on ρ but independent of j, such that

$$\prod_{k=0}^{j-1} |z_j - z_k| \leq \gamma_2 \rho^j, \quad j \geq 1. \tag{2.16}$$

Substitution of (2.16) into (2.15) yields the bound

$$\|T_2\| \leq \gamma_2 \sum_{j=0}^{n} \rho^j \leq \gamma_2(\rho - 1)^{-1} \rho^{n+1}. \tag{2.17}$$

Since $\rho > 1$ in (2.17) can be chosen arbitrarily close to 1, we obtain

$$\overline{\lim_{n\to\infty}} \|T_2\|^{1/n} \leq 1. \tag{2.18}$$

From $\|T\| \leq \|T_1\| \, \|T_2\|$ and the bounds (2.14) and (2.18), it follows that

$$\overline{\lim_{n\to\infty}} \|T\|^{1/n} \leq 1. \tag{2.19}$$

It is easy to bound the inverse mapping $T^{-1}p_n = \mathbf{f}$, where $p_n \in \Pi_{n,\rho,M}$ is given by (1.4) and $\mathbf{f} = (f_0, f_1, \ldots, f_n)^T \in C^{n+1}_{\rho,M}$. From $p_n(z_j) = f_j$ for $0 \leq j \leq n$, we obtain

$$\|T^{-1}\| = \max_{\|p_n\|_{\partial K} = 1} \|\mathbf{f}\|_\infty \leq 1. \tag{2.20}$$

By (2.19)-(2.20) it follows that

$$\overline{\lim_{n\to\infty}} \operatorname{cond}(T)^{1/n} \leq 1. \tag{2.21}$$

Now (2.10) follows from (2.21) and from the inequality $\operatorname{cond}(T) = \|T\| \, \|T^{-1}\| \geq 1$.

We have so far assumed that the capacity of K is 1. However, (2.10) can be shown independently of the scaling. If K has capacity different from 1, then $\|T_1\|$ and $\|T_2\|$ grow or decrease exponentially as n increases, and the proof has to be modified slightly. ∎

We remark that if K has capacity different from 1, then the exponential growth or decrease of $\|T_1\|$ and $\|T_2\|$ as n increases can give rise to numerical difficulties, related to overflow or underflow. These difficulties stem from the fact that the products

$$\prod_{\substack{j=0 \\ j\neq k}}^{n} |z_k - z_j| \text{ and } \prod_{j=0}^{n-1} |z - z_j|, \qquad z \in \partial K,$$

may grow or decrease exponentially with n; see Lemmas 2.1-2.2. We therefore scale K to have capacity 1 or close to 1 in the computed examples of Section 4.

3. Newton interpolation at arbitrary boundary points.

Let the set K satisfy the conditions of the first paragraph of Section 1, and assume that f is analytic on K. Let S_m be a set of m distinct points on ∂K, and let $\{z_j\}_{j=0}^{m-1}$ denote a sequence of Leja points for S_m. This section discusses how to approximate f on K by polynomials $p_n \in \Pi_n$, $n < m$, that interpolate f at the points $z_0, z_1, \ldots, z_n$.

Assume that the points of S_m are distributed fairly uniformly over all of ∂K. Then we expect that for some (possibly small) value of $l < m$, the sets $\{z_j\}_{j=0}^{n}$, $0 \leq n \leq l$, of Leja points for S_m are good approximations of sets of Leja points for K, in the sense that

$$\left\| \prod_{j=0}^{n-1} (z - z_j) \right\|_{S_m}^{1/n} \approx \left\| \prod_{j=0}^{n-1} (z - z_j) \right\|_{\partial K}^{1/n}, \quad 0 \leq n \leq l, \tag{3.1}$$

where we note that the left-hand side of (3.1) is never larger than the right-hand side. Formula (3.1) suggests that the error $\|f - p_n\|_{\partial K}$ should decrease geometrically as n increases, so long as n is sufficiently small in relation to m. For larger values of n, the right-hand side of (3.1) might be much larger than the left-hand side, and the error $\|f - p_n\|_{\partial K}$ might increase with n. This behavior of $\|f - p_n\|_{\partial K}$ can be observed in computed examples of Section 4, and suggests the following numerical scheme for the approximation of f by p_n on K.

1. If the capacity of K is explicitly known, then scale K to have capacity 1. Compute (the first few) Leja points $z_0, z_1, z_2 \ldots$ for S_m.
2. If the capacity of K is known, then goto Step 3. Otherwise compute

$$n \to \prod_{j=0}^{n-1} |z_n - z_j|^{1/n} \tag{3.2}$$

   for increasing values of n in order to determine an estimate for the capacity of K. In view of Lemma 2.1 and (3.1), the expression (3.2) yields an estimate $\tilde{c}$ for the capacity of K if n is sufficiently large and $n \leq l$. In many applications it suffices to evaluate (3.2) for fairly small values of n. Replace K by $\tilde{c}^{-1}K$. The latter set has capacity approximately 1.

3. Compute $n \to \|f - p_n\|_d$ for increasing values of n and select the polynomial of lowest degree that yields a sufficiently small approximation error. Here $\| \; \|_d$ denotes a discrete maximum seminorm on a fine mesh on ∂K.

The scheme above may require the determination of more Leja points for S_m if the accuracy achieved in Step 3 is insufficient. The scheme is simple and fast when carried out interactively, e.g., at a graphics terminal or workstation.

Finally, we remark that the requirement that the points of S_m be distinct can be removed. Let S_m contain the distinct point $z_0, z_1, \ldots, z_{l-1}$, and let $\mu(z_k)$ denote the multiplicity of $z_k \in S_m$. We interpolate $f(z), f^{(1)}(z), \ldots, f^{(\mu(z_k)-1)}(z)$ at $z = z_k$, where $f^{(j)}$

denotes the jth derivative of f. An ordering of the distinct points $z_k \in S_m$ is obtained by requiring these points to satisfy (1.5a) and

$$\text{(1.5b')} \quad \prod_{k=0}^{j} |z_{j+1} - z_k|^{\mu(z_k)} = \max_{z \in S_m} \prod_{k=0}^{j} |z - z_k|^{\mu(z_k)}, \qquad z_{j+1} \in S_m, \quad 0 \le j < l.$$

4. Numerical examples.

The examples of this section have been computed on an IBM 3090VF computer in single precision arithmetic, i.e. with only about six significant digits. We used the program of de Boor [3, p. 24], after simple modifications, to compute and evaluate the interpolation polynomials in Newton form. The purpose of the modifications was to allow i) different interpolation points, ii) the ordering (1.5) of the interpolation points, iii) complex arithmetic, and iv) interpolation polynomials of higher degree.

EXAMPLE 4.1. This example illustrates the stability that can be achieved when ordering the interpolation points according to (1.5), and when scaling K to have capacity 1. For comparison, we first consider a straightforward ordering of z_j, and a different scaling of K.

Approximate $f(z) := (1 + z)^{1/2}$ on $K := [-1, 1]$ by the polynomial $p_n \in \Pi_n$ in Newton form (1.4) that interpolates f at the zeros of the Chebyshev polynomial $T_{n+1}(z) := \cos((n + 1)\arccos(z))$. We order the interpolation points z_j from right to left, i.e. we use the points

$$\text{(4.1)} \qquad z_j := \cos\left(\frac{2j + 1}{2(n + 1)}\pi\right), \qquad 0 \le j \le n.$$

The error $f - p_n$ is evaluated at 19 equidistant nodes between each pair of adjacent interpolation points. We denote this seminorm by $\|f - p_n\|_d$ and obtain Table 4.1.

Table 4.1. *Interpolation of* $f(z) = (1 + z)^{1/2}$ *at points* (4.1)

n	$\|f - p_n\|_d$
1	$0.7925 \cdot 10^{-1}$
3	$0.3005 \cdot 10^{-1}$
5	$0.1905 \cdot 10^{-1}$
7	$0.1404 \cdot 10^{-1}$
9	$0.1114 \cdot 10^{-1}$
11	$0.9242 \cdot 10^{-2}$
13	$0.7900 \cdot 10^{-2}$
15	$0.8811 \cdot 10^{-2}$
17	$0.6336 \cdot 10^{-1}$
19	0.2804

Table 4.1 shows that propagated round-off errors give rise to large errors in p_n for

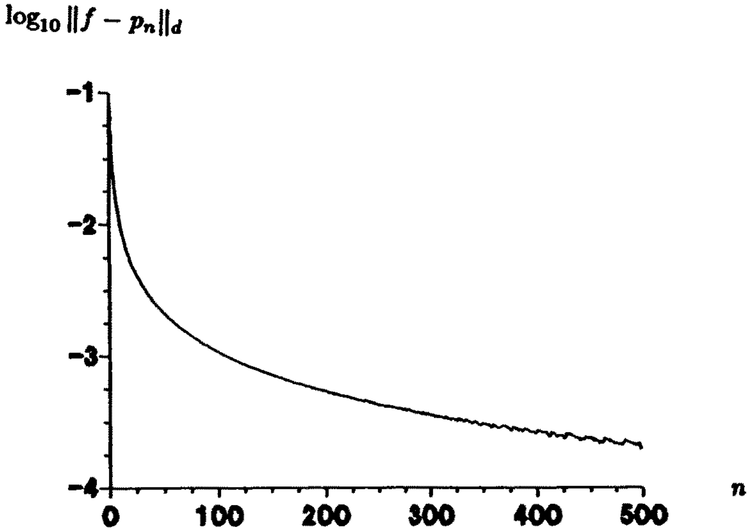

Fig. 4.1. Interpolation fo $f(z) = (1 + z/2)^{1/2}$ at Leja points for (4.2).

$n \geq 15$. The rapid growth of the propagated round-off error with n depends on the condition number cond(T) growing exponentially and rapidly with n. The computations for Table 4.1 have previously been carried out by de Boor [3, p. 28]. He suggests [3, p. 36] reordering of the points (4.1) as a possible way to reduce the propagated error, but does not present an algorithm for the reordering.

We now scale the interval to have capacity one, and also change the ordering of the interpolation points, i.e. we consider the approximation of $f(z) := (1 + z/2)^{1/2}$ on the interval $K := [-2, 2]$ by a polynomial $p_n \in \Pi_n$ of the form (1.4), that interpolates f at Leja points for

$$S_{n+1} := \left\{2\cos\left(\frac{2j+1}{2(n+1)}\pi\right) : 0 \leq j \leq n\right\}. \tag{4.2}$$

The error $\|f - p_n\|_d$ is computed analogously as for Table 4.1. Figure 4.1 shows this error for $1 \leq n \leq 500$, and demonstrates the stability of the scheme. We remark that the scaling of K has no effect on cond(T). The scaling is carried out in order to avoid overflow and malign underflow during the computations.

EXAMPLE 4.2. Consider the approximation of $f(z) := (1 + 6.25z^2)^{-1}$ on $K := [-2, 2]$ by polynomials. Introduce sets of equidistant points on K,

$$S_m := \{-2 + 4j(m-1)^{-1} : 0 \leq j \leq m\}, \qquad m \leq 2.$$

It is well known that the polynomials $q_{m-1} \in \Pi_{m-1}$ that interpolate f at the points of S_m do not converge to f on K as m increases; see, e.g., [3, pp. 24–26] or [4, pp. 101–102]. A simple way to compute polynomial approximants of functions tabulated at the points of S_m for some $m \geq 2$ is to interpolate at Leja points

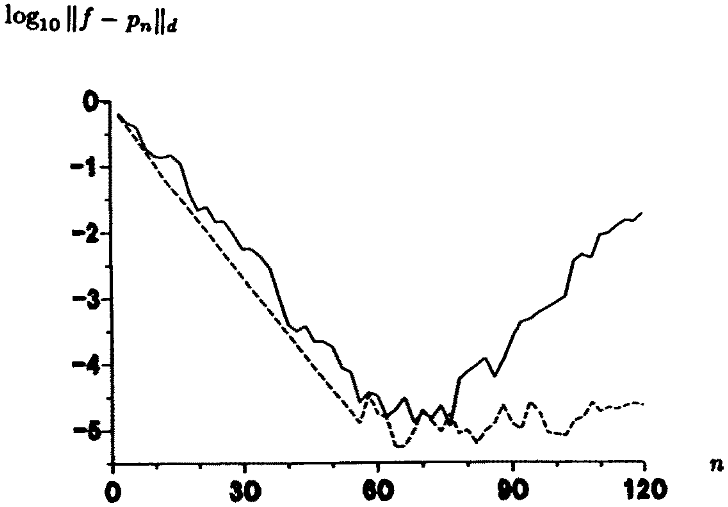

Fig. 4.2. Interpolation of $f(z) = (1 + 6.25z^2)^{-1}$ at Leja points of S_{500} (continuous curve) and at Chebyshev points (dashed curve). Error shown for polynomials of even degree only.

$z_0, z_1, \ldots, z_n$ of S_m, and increase the degree n of the interpolation polynomial p_n given by (1.4) until a sufficiently small approximation error is obtained. The ease with which p_{n+1} can be computed from p_n, and z_{n+1} can be determined from $z_0, z_1, \ldots, z_n$, makes this scheme attractive. Figure 4.2 illustrates results of this scheme when f is tabulated at the points of S_{500}. The error $f - p_n$ is measured by the seminorm $\|g\|_d := \max_{0 \le j \le 10^4} |g(-2 + j4 \cdot 10^{-4})|$ and the *continuous curve* shows $\log_{10} \|f - p_{2j}\|_d$ for $0 \le j \le 60$. The approximation error $\|f - p_{2j}\|$ was generally found to be smaller than the errors $\|f - p_{2j+1}\|_d$ and $\|f - p_{2j-1}\|_d$. We therefore only show the error for interpolation polynomials of even degree.

The *dashed curve* of Figure 4.2 is included for comparison and shows $\|f - q_{2j}\|_d$ for $1 \le j \le 60$, where q_{2j} is the polynomial that interpolates f at the Leja points for $S_{2j+1} := \{2\cos\left(\frac{2k+1}{4j+2}\pi\right),\ 0 \le k \le 2j\}$. Hence, the error $\|f - q_{2j}\|_K$ is very close to the error $\min_{p \in \Pi_{2j}} \|f - p\|_K$; see de Boor [3, Chap. 2] for a discussion. Figure 4.2 shows the geometric convergence of $\|f - p_{2j}\|_d$ for increasing values of j, for $j \le 25$. It also shows that the error $\|f - p_{2j}\|_d$ is fairly close to $\|f - q_{2j}\|_d$, and therefore to $\min_{p \in \Pi_{2j}} \|f - p\|_d$, for $j \le 35$. For large values of j the error $\|f - p_{2j}\|_d$ increases exponentially with j, in agreement with the discussion of Section 3.

EXAMPLE 4.3. This example shows how interpolation at Leja points can be applied to determine polynomials that approximate analytic functions f on regions K whose capacity is not explicitly known or difficult to compute. We illustrate the scheme for $K := \{z = x + iy : -1 \le x \le 1, -\frac{1}{2} \le y \le \frac{1}{2}\}$ and $f(z) := (2.1 - z)^{-1}$, and assume that we do not know the capacity of K.

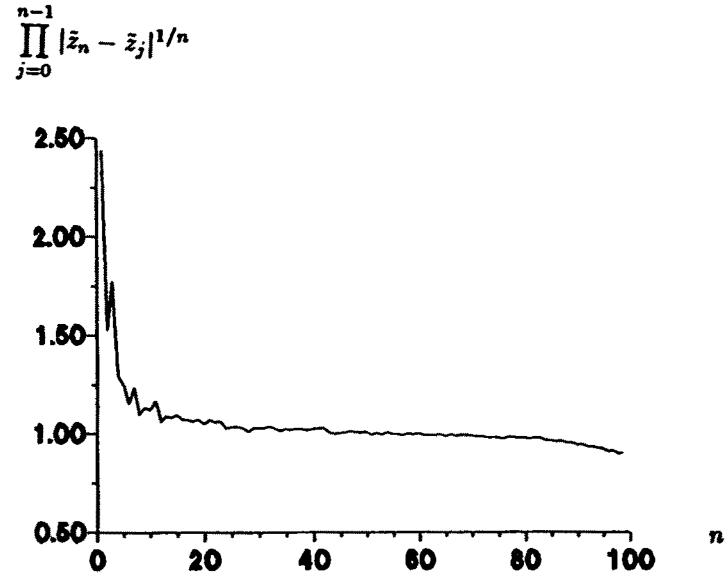

Fig. 4.3. Approximate capacity of $\tilde{K}$.

Let $z(t), 0 \leq t < 6$, be a parametric representation of ∂K with respect to arc length. We choose $z(0) = 1$ and traverse ∂K in the counter-clockwise direction as t increases. Let $S_{100} := \{z(6j/100), \quad 0 \leq j < 100\}$, and let $z_0, z_1, z_2, \ldots$ denote Leja points for S_{100}. An approximation of the capacity of K is obtained by evaluating $\prod_{j=0}^{n-1} |z_n - z_j|^{1/n}$ for increasing values of n; see Section 3. Doing so shows that K has capacity close to 0.9. We therefore introduce $\tilde{K} := 1.1K := \{1.1z : z \in K\}$ and $\tilde{S}_{100} := 1.1S_{100}$. The points $\tilde{z}_j := 1.1z_j$ are Leja points for $\tilde{S}_{100}$. Figure 4.3 shows $\prod_{j=0}^{n-1} |\tilde{z}_n - \tilde{z}_j|^{1/n}$ for increasing values of n and indicates that $\tilde{K}$ has capacity close to 1. (For the present set K the capacity can be computed by evaluating an elliptic function; see Bickley [1, p. 86]. Hough and Papamichael [8, p. 303] have in this manner determined that the capacity of $2K$ is 1.7495146. Thus, the capacity of K is

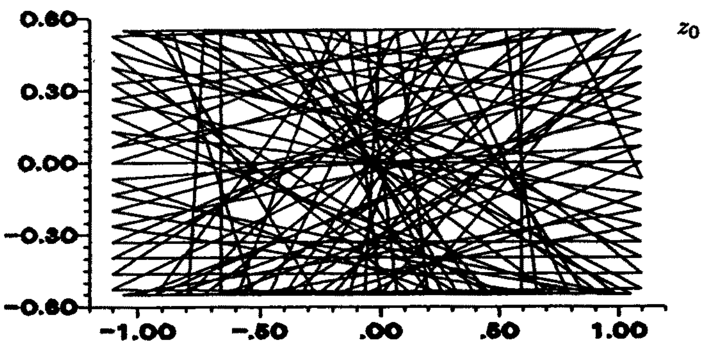

Fig. 4.4. Edges between successive Leja points $\tilde{z}_j$.

approximately 0.875, and the capacity of $\tilde{K}$ is approximately 0.962. This shows that the approximation of the capacity found by computing $\prod_{j=0}^{n-1} |\tilde{z}_n - \tilde{z}_j|^{1/n}$ for increasing values of n is fairly accurate. On the other hand, high accuracy cannot be expected even if the points z_j were Leja points for K; see Gaier [6, pp. 178–179] for a discussion.)

The distribution of the Leja points $\tilde{z}_j$ is illustrated by Figure 4.4, which shows the edges connecting each pair of points $\{\tilde{z}_j, \tilde{z}_{j+1}\}$ for $0 \le j \le 100$. Figure 4.4 indicates that successive Leja points are not close. The point $\tilde{z}_0$ is located in the upper-right corner.

Let p_n be the Newton interpolation polynomial (1.4a) that interpolates f at the points $\tilde{z}_0, \tilde{z}_1, \ldots, \tilde{z}_n$. We measure the error $f - p_n$ by the seminorm $\|g\|_d := \max_{0 \le j < 2000} |g(z(j \cdot 0.003))|$. Figure 4.5 shows $\|f - p_n\|_d$ and illustrates that the error decreases geometrically as n increases and is sufficiently small.

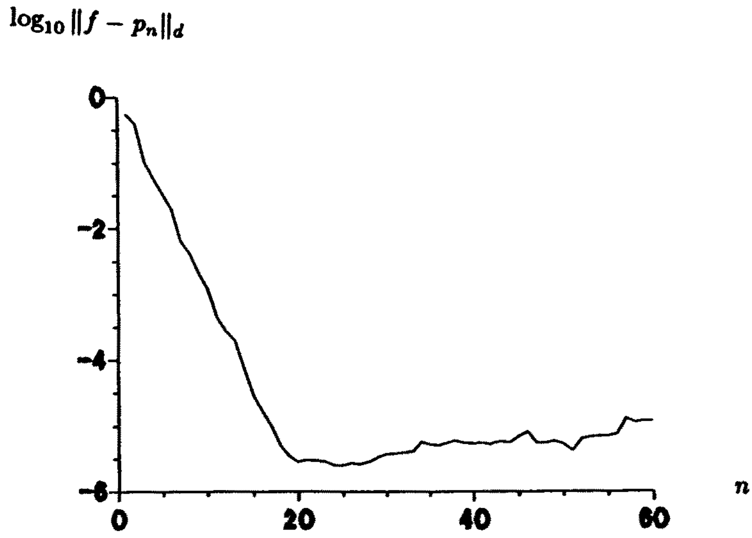

Fig. 4.5. Interpolaton of $f(z) = (2.1 - z)^{-1}$ at Leja points for $\tilde{S}_{100}$.

5. Conclusion.

The condition number of the Newton form defined by interpolation at Leja points is shown to grow slower than exponentially with the number of interpolation points. A simple, fairly stable scheme for polynomial approximation of analytic functions on compact point sets in the complex plane is obtained by interpolation at Leja points for a finite subset of the boundary points.

Acknowledgements.

This paper was completed during a visit at M.I.T., Department of Mathematics. I would like to thank Prof. L. N. Trefethen for his kind hospitality and for comments on a preliminary version of this paper. I also would like to thank Prof. D. Gaier for comments and references.

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