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The Evaluation of Polynomials

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1. Introduction

In evaluating the polynomial

$$P(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n, \qquad (a_i \text{ real}) \tag{1}$$

by the Newton-Horner recurrence scheme,

$$P_0 = a_0,$$
$$P_{i+1} = x P_i + a_{i+1}, \qquad i = 0, 1, 2, \ldots, n-1$$

n multiplications and n additions are required. OSTROWSKI [5] has shown that the $2n$ operations required by this algorithm are minimal for $n \leqq 4$. MOTZKIN [4] (see also TODD [8]) and KNUTH [3] have given methods whereby polynomials with $4 \leqq n \leqq 6$ can be evaluated in $[\frac{1}{2}(n+3)]$ multiplications and not more than $n+1$ additions. Similar methods effecting a reduction in the number of multiplications have been described by PAN [6] for $n \leqq 12$. Each of these methods is valid only for a particular value of n.

A general method due to PAN [7] applicable to all polynomials with $n \geqq 5$ results in an evaluation involving $[\frac{1}{2}(n+4)]$ multiplications and $n+1$ additions. KNUTH has also given a method applicable to all polynomials with $n \geqq 3$ in which $n+1$ additions are required while the number of multiplications varies between $[\frac{1}{2}(n+3)]$ and approximately $\frac{3}{4}n$.

BELAGA [1] has shown that at least n additions are necessary in evaluating an arbitrary n-th degree polynomial and consideration of the simple examples $a_0 x^3$, $a_0 x^4$ etc. indicates a lower bound on the number of multiplications of $[\frac{1}{2}(n+3)]$. BELAGA also proves that $[\frac{1}{2}(n+3)]$ multiplications and $n+1$ additions will always suffice to evaluate an n-th degree polynomial but these counts may include complex multiplications and additions.

It will be shown that Knuth's method may be modified to produce an evaluation process requiring $[\frac{1}{2}(n+4)]$ multiplications and n additions. Clearly for n odd such an evaluation involves the minimum number of operations but when n is even it is still uncertain whether a further multiplication can be eliminated. Knuth's special methods for degrees 4 and 6 achieve this at the expense of an extra addition but no such method is known for degree 8.

2. Knuth's Method

Expressing $P(x)$ in the form

$$P(x) = Q_0(x) + Q_1(x),$$

where

$$Q_0(x) = a_0 x^n + a_2 x^{n-2} + a_4 x^{n-4} + \cdots$$

$$Q_1(x) = a_1 x^{n-1} + a_3 x^{n-3} + a_5 x^{n-5} + \cdots$$

those features of Knuth's method which will be required may be summarised as follows.

When $n = 2m + 2$ is even, polynomials S_0 and T are defined by

$$S_0(x^2) = Q_0(x),$$

$$T(x^2) = Q_1(x)/x$$

while when $n = 2m + 1$ is odd, these polynomials are defined by

$$S_0(x^2) = Q_1(x),$$

$$T(x^2) = Q_0(x)/x.$$

If the m zeros of $T(x)$ are denoted by $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m$ a sequence of constants $\beta_m, \beta_{m-1}, \ldots$ can be determined from $S_0(x^2)$ by repeated synthetic division by the quadratic factors $(x^2 - \alpha_i)$ according to the scheme

$$S_{i-1}(x^2) = (x^2 - \alpha_{m-i+1})\, S_i(x^2) + \beta_{m-i+1}, \qquad i = 1, 2, \ldots, m$$

where finally

$$\begin{aligned} S_m(x^2) &= a_0 x^2 + \beta_0 && \text{if } n \text{ is even} \\ &= a_1 && \text{if } n \text{ is odd.} \end{aligned}$$

It follows immediately that

$$P(x) = ((\ldots((P_1(x^2 - \alpha_1) + \beta_1)(x^2 - \alpha_2) + \beta_2)\ldots)(x^2 - \alpha_m) + \beta_m), \tag{2}$$

where

$$\begin{aligned} P_1 &= a_0 x^2 + a_1 x + \beta_0 && \text{if } n \text{ is even} \\ &= a_0 x + a_1 && \text{if } n \text{ is odd.} \end{aligned}$$

Using these values of P_1, (2) can be evaluated by

$$P_{i+1} = (x^2 - \alpha_i)\, P_i + \beta_i, \qquad i = 1, 2, \ldots, m - 1,$$

$$P(x) = (x^2 - \alpha_m)\, P_m + \beta_m,$$

which requires $[\frac{1}{2}(n + 4)]$ multiplications and n additions. This operation count implies making one of two equally untenable assumptions, either that the α_i are always real, or that complex multiplications and additions should be counted with the same weight as their real counterparts.

KNUTH also points out that by a shift of origin i.e. a transformation

$$y = x + c$$

any polynomial in y can be transformed to the form (1) in which by a suitable choice of c, $a_1 = a_0$. This transformation has the advantage when n is even that

$$P_1 = a_0(x^2 + x) + \beta_0,$$

so that a multiplication can be saved at the expense of the extra addition for the change of variable. With this device included, Knuth's evaluation algorithm for even degree polynomials is equivalent to that used by BELAGA in his proof that any polynomial may be evaluated in $[\frac{1}{2}(n+3)]$ multiplications and $n+1$ additions.

3. Modification of Knuth's Method

The following theorem justifies a transformation which converts an arbitrary polynomial into one producing real α_i.

Theorem. *The m zeros of the polynomial $T(x)$ are all real if at least $n-1$ zeros of $P(x)$ are confined to one of the half-planes* $\mathrm{Re}(x) \geqq 0$ *or* $\mathrm{Re}(x) \leqq 0$.

$Q_0(x)$ and $Q_1(x)$ have real coefficients and each involves only odd powers of x or only even powers of x so that $Q_0(x)$ and $Q_1(x)$ cannot have less than four complex (as opposed to real or imaginary) zeros. The theorem is established by showing that $Q_0(x)$ and $Q_1(x)$ have at least $n-3$ imaginary zeros, under the conditions of the theorem, since it now follows that the remaining zeros of $Q_0(x)$ and $Q_1(x)$ are real or purely imaginary; hence the zeros of $T(x)$ being the squares of zeros of $Q_0(x)$ or $Q_1(x)$ are real.

Let $P(x)$ have k zeros in the half-plane $\mathrm{Re}(x) > 0$.

Case 1. $P(x)$ has no zeros on $\mathrm{Re}(x)=0$. In this case one way of expressing Routh's rule for determining k is as follows (see GANTMACHER [2]). As x traverses the real axis from $-\infty$ to $+\infty$, the number of transitions from $-\infty$ to $+\infty$ minus the number of transitions from $+\infty$ to $-\infty$ of the rational function

$$R(x) = i\,Q_1(i\,x)/Q_0(i\,x)$$

is $n-2k$. i.e. the Cauchy Index $I_{-\infty}^{\infty} R(x) = n-2k$.

For $k=0$ or n, this implies that $Q_0(x)$ has n imaginary zeros. For $k=1$ or $n-1$, $Q_0(x)$ must have at least $n-2$ imaginary zeros.

Using the fact that $R(x)$ is an odd function of x with one of the properties of the Cauchy Index given by GANTMACHER, it is readily established that

$$I_{-\infty}^{\infty} R^{-1}(x) + I_{-\infty}^{\infty} R(x) = \sigma,$$

where $\sigma = +1$ or -1 according to whether $R(x) > 0$ or $R(x) < 0$ as $x \to +\infty$.

Consequently $I_{-\infty}^{\infty} R^{-1}(x) = 2k + \sigma - n$ and for $k = 0, 1, n-1, n$

$$|2k + \sigma - n| \geqq n-3,$$

so that $Q_1(x)$ has at least $n-3$ imaginary zeros.

Case 2. $P(x)$ has t zeros on $\mathrm{Re}(x)=0$, $0 < t < n-3$. (The case $t \geqq n-3$ is trivial.) The t zeros of $P(x)$ on $\mathrm{Re}(x)=0$ are also zeros of $Q_0(x)$ and $Q_1(x)$. Dividing out the factors corresponding to these zeros from $P(x)$, $Q_0(x)$ and $Q_1(x)$, the reduced polynomials resulting $P'(x)$, $Q_0'(x)$ and $Q_1'(x)$ can be treated by Case 1, showing that $Q_0'(x)$ and $Q_1'(x)$ have at least $n-t-3$ imaginary zeros so that $Q_0(x)$ and $Q_1(x)$ again possess at least $n-3$ imaginary zeros.

Application of the Theorem. An arbitrary polynomial in the variable y may be transformed using

$$x = y + c$$

to the form (1). The constant c is chosen such that at least $n-1$ zeros of the new polynomial $P(x)$ lie in one of the half-planes $\mathrm{Re}(x) \geqq 0$ or $\mathrm{Re}(x) \leqq 0$ and at least two zeros are symmetrically disposed about the origin. Such a pair of zeros produce a factor $(x^2-\alpha)$ of $P(x)$ (and consequently of $T(x^2)$ and $S_0(x^2)$). Taking this value of α to be α_m, since $\beta_m = 0$ the evaluation algorithm now becomes

$$\begin{aligned} x &= y + c, \\ P_1 &= a_0 x^2 + a_1 x + \beta_0 \quad \text{if} \quad n = 2m+2 \\ &= a_0 x + a_1 \qquad\qquad \text{if} \quad n = 2m+1 \\ P_{i+1} &= (x^2-\alpha_i)\, P_i + \beta_i, \quad i = 1, 2, \ldots, m-1 \\ P(x) &= (x^2-\alpha_m)\, P_m, \end{aligned} \tag{3}$$

which requires n additions and $[\frac{1}{2}(n+4)]$ multiplications.

It is clear that for any polynomial several different sets of values exist for the constants α_i, β_i and c. Confining $n-1$ zeros to the right or left half-plane results in different sets of constants; often there exists a choice between shifting the origin to lie mid-way between two real zeros or so that a complex conjugate pair of zeros "between" these real zeros becomes purely imaginary. Permuting the labelling of the α_i changes the values of the β_i. In a practical problem the choice will be influenced by consideration of the accuracy of the evaluation.

The process is illustrated with the obviously contrived example

$$P(y) = 2y^8 + 160y^6 - 1040y^5 + 3268y^4 - 17760y^3 + 15120y^2 - 32400y - 69750$$

for which $T(y^2)$ does not exist. (The y^7 term is absent.) The zeros of largest real part of $P(y)$ are a real zero at $y=5$ and a complex conjugate pair at $y = 1 \pm 2i$. Taking out the factor corresponding to this complex conjugate pair and changing the variable using

$$y = x + 1,$$

so that this pair become imaginary, the equivalent polynomial is

$$P(x) = (2x^6 + 16x^5 + 208x^4 - 32x^3 - 224x^2 - 11648x - 25600)\,(x^2+4).$$

Then

$$T(x^2) = (16x^4 - 32x^2 - 11648)\,(x^2+4) = 16\,(x^2-28)\,(x^2+26)\,(x^2+4).$$

Taking $\alpha_1 = 28$, $\alpha_2 = -26$ and $\alpha_3 = -4$, β_2 is obtained on dividing

$$S_1(x^2) = 2x^6 + 208x^4 - 224x^2 - 25600$$

by $x^2 - \alpha_2$ obtaining

$$S_2(x^2) = 2x^4 + 156x^2 - 4280 \quad \text{and} \quad \beta_2 = 85680.$$

Dividing $S_2(x^2)$ by $x^2 - \alpha_1$ yields

$$S_3(x^2) = 2x^2 + 212 \quad \text{and} \quad \beta_1 = 1656.$$

The constant term in $S_3(x^2)$ is $\beta_0 = 212$. $a_0 = 2$ and $a_1 = 16$ are the leading coefficients of $P(x)$; the constant c in the transformation is -1 so that all quantities required in the evaluation algorithm (3) have been determined.

References

[1] Belaga, E. G.: Some problems involved in the calculation of polynomials. Dok. Akad. Nauk SSSR. **123**, 775 (1958).
[2] Gantmacher, F. R.: The Theory of Matrices, Vol. II, Ch. XV and p. 216. Chelsea, N.Y. 1959.
[3] Knuth, D. E.: Evaluation of polynomials by computer. C.A.C.M. **5**, 595 (1962).
[4] Motzkin, T. S.: Evaluation of polynomials. Bull. Amer. Math. Soc. **61**, 163 (1955).
[5] Ostrowski, A. M.: Studies in mathematics and mechanics, p. 44. Presented to R. von Mises. New York: Academic Press 1954.
[6] Pan, V. Ya.: Certain schemes for the calculation of values of polynomials with real coefficients. Problemy Kibernetiki **5**, 17 (1959).
[7] — Schemes for the calculation of polynomials with real coefficients. Dok. Akad. Nauk SSSR. **127**, 266 (1959).
[8] Todd, J.: Motivation for working in numerical analysis. Comm. Pure Appl. Math. **8**, 97 (1955).

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