On continuous dependence of roots of polynomials on coefficients

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Abstract

It is a well know result that roots of a polynomial depend continuously on its coefficients. Here we review this basic result and produce a proof via the use of Rouché's Theorem. We also provide a simple result regarding real simple roots of polynomials with real coefficients.

1 Introduction

The basic result we would like to discuss here is that of continuous dependence of of roots of a polynomial on its coefficients. First, we discuss the requisite complex analysis basics. Next, using Rouché's Theorem, we provide a proof of the Fundamental Theorem of Algebra and then the result on continuous dependence of polynomials on their coefficients. We also briefly discuss simple real roots of polynomials with real coefficients at the end of this note, where we provide a simple proof of the following result: in addition to continuous dependence, real simple roots of a polynomials with real coefficients remain real under sufficiently small (real) perturbation to the coefficients.

2 Zeros of analytic functions

Recall that a function $f : \mathbb{C} \to \mathbb{C}$ is called analytic in an open set $U \subset \mathbb{C}$ if it has a derivative at each point in U. Let us begin by considering an analytic function f defined on a simply connected domain $U \subseteq \mathbb{C}$. Let C be a simple closed contour in U. We denote by $N_z(f; C)$ the number of zeros of f in the interior of C. The following basic result [1, 2] from complex analysis, which is a simple application of the Residue Theorem, allows one to compute $N_z(f; C)$.

Theorem 2.1. *Let f be an analytic function on a simply connected domain $U \subseteq \mathbb{C}$. Let C be a positively oriented simple closed contour in U such that $f(z) \neq 0$ on C. Then, f has finitely many zeros in the interior of C given by,*

$$N_z(f; C) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz.$$

The following result, due to Rouché, is central to the discussion in this note. The presented proof follows that of [1] closely.

Theorem 2.2 (Rouché)**.** *Let f and g be analytic in a simply connected domain U. Let C be a simple closed contour in U. If $|f(z)| > |g(z)|$ for every z on C, then the functions $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, inside C.*

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Proof. We begin by defining the following function,

$$\Psi(t) = \frac{1}{2\pi i} \int_C \frac{f'(z) + tg'(z)}{f(z) + tg(z)}\, dz, \quad t \in [0,1].$$

Note that by Theorem 2.1, $N_z(f;C) = \Psi(0)$ and $N_z(f+g;C) = \Psi(1)$. First, let us note that the denominator of the the integrand in the definition of Ψ is never zero. This is seen by noting that for any $z \in C$,

$$|f(z) + tg(z)| \geq ||f(z)| - t|g(z)|| \geq |f(z)| - |g(z)| > 0,$$

It is also straightforward to see that the function Ψ is continuous on $[0,1]$. Moreover, by Theorem 2.1, Ψ is integer valued. Therefore, it must be the case that Ψ is constant on $[0,1]$. In particular, it follows that

$$N_z(f;C) = \Psi(0) = \Psi(1) = N_z(f+g;C).$$

□

3 Roots of polynomials

3.1 Fundamental Theorem of Algebra

The following proof of the Fundamental Theorem of Algebra uses Rouché's Theorem. The short argument for this fundamental result shows the power of Rouché's theorem. The proof presented here is standard and follows that of [4] closely. In the following, when discussing a polynomial of degree n,

$$p(z) = \sum_{k=0}^{n} a_k z^k, \quad a_k \in \mathbb{C},$$

we lose no generality if we consider the case when $a_n = 1$; that is, we consider,

$$p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k, \quad a_k \in \mathbb{C}. \tag{3.1}$$

Theorem 3.1 (Fundamental Theorem of Algebra)**.** *Let $p(z)$ be a polynomial of degree n, $n \geq 1$, with complex coefficients as in* (3.1)*. Then, p has exactly n zeros counting multiplicities.*

Proof. Let $f(z) = z^n$, and note that trivially f has n zeroes (counting multiplicity). Next, define $g(z) = \sum_{k=0}^{n-1} a_k z^k$, where a_k, $k = 0, \ldots, n-1$, are the coefficients of p as in (3.1). Choose $R > 0$ such that $R > 1 + \sum_{k=0}^{n-1} |a_k|$, and let C_R be the circle $\{z : |z| = R\}$. For each $z \in C_R$, we have,

$$|g(z)| = |\sum_{k=0}^{n-1} a_k z^k| \leq \sum_{k=0}^{n-1} |a_k||z^k| = \sum_{k=0}^{n-1} |a_k| R^k < \sum_{k=0}^{n-1} |a_k| R^{n-1} < RR^{n-1} = R^n = |z|^n = |f(z)|.$$

That is $|f(z)| > |g(z)|$ on the circle C_R. Recalling that f has n zeros, it follows by Rouché's Theorem that $f(z) + g(z) = p(z)$ also has n zeros inside C_R. Since R can be made arbitrarily large, it follows that $p(z)$ has exactly n zeros. □

3.2 Continuous dependence of roots of polynomials on coefficients

We know by the Fundamental Theorem of Algebra that a polynomial of degree n has n (complex) roots. Theorem 3.2 below, which is the main point of the discussion in this note, shows that the roots of a polynomial depend continuously to its coefficients; its proof is an adaptation of the argument presented in [3, p. 43].

In the following we denote by $B(z_0, \varepsilon)$ the open ball of radius ε centered at $z_0 \in \mathbb{C}$:

$$B(z_0, \varepsilon) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}.$$

Theorem 3.2. *Let $p(z)$ be a polynomial of degree n, $n \geq 1$, as in* (3.1)*, with m distinct roots $\{\lambda_1, \ldots, \lambda_m\}$, $1 \leq m \leq n$, having multiplicities $\alpha_1, \ldots, \alpha_m$, Then, for any $\varepsilon > 0$ such that the closed balls $\overline{B(\lambda_j, \varepsilon)}$, $j = 1, \ldots, m$ are disjoint, there exists a $\delta = \delta(\varepsilon) > 0$ such that any degree n polynomial,*

$$q(z) = z^n + \sum_{k=0}^{n-1} b_k z^k,$$

with $|a_k - b_k| < \delta$, $k = 0, \ldots, n-1$, has exactly α_j roots (counting multiplicities) in $B(\lambda_j, \varepsilon)$, $j = 1, \ldots, m$.

Proof. Let ε be as in the statement of the theorem, and define the circles

$$C_j = \{z : |z - \lambda_j| = \varepsilon\}, \quad j = 1, \ldots, m;$$

recall that by the assumption on ε, the circles C_j are disjoint. Let

$$\nu_j = \min_{z \in C_j} |p(z)|, \quad j = 1, \ldots, m,$$

where we know the minimums ν_j are attained because for each j, we have a continuous function on a compact set. Moreover, $\nu_j > 0$, because the zeros of p lie at the centers of the disjoint circles C_j, $j = 1, \ldots, m$. Next, let ρ_j, $j = 1, \ldots, m$ be defined as follows,

$$\rho_j = \max_{z \in C_j} \{1 + \sum_{k=1}^{n-1} |z^k|\}.$$

Choose $\{b_k\}_0^{n-1}$ such that $|a_k - b_k| < \delta$ with $\delta > 0$ taken such that,

$$\delta \rho_j < \nu_j, \quad j = 1, \ldots, m.$$

Then, we have for $z \in C_j$,

$$\begin{aligned} |q(z) - p(z)| &= \Big| \sum_{k=0}^{n-1} (b_k - a_k) z^k \Big| \\ &\leq \sum_{k=0}^{n-1} |b_k - a_k| |z^k| \\ &< \delta \sum_{k=0}^{n-1} |z^k| \leq \delta \rho_j < \nu_j \leq |p(z)|. \end{aligned}$$

That is, for $j = 1, \ldots, m$,

$$|p(z)| > |q(z) - p(z)|, \quad \text{on } C_j.$$

Therefore, it follows by Rouché's Theorem that $p(z)$ and $(q(z) - p(z)) + p(z) = q(z)$ have the same number of zeros, namely α_j, (counting multiplicities) in the interior of C_j, i.e. $B(\lambda_j, \varepsilon)$, for each $j = 1, \ldots, m$. □

Let $p(z)$ be a polynomial of degree n, $n \geq 1$, as in (3.1), with m distinct roots $\{\lambda_1, \ldots, \lambda_m\}$, $1 \leq m \leq n$. Define $R_0(p)$ as follows,

$$R_0(p) = \begin{cases} \frac{1}{2}, & \text{if } m = 1, \\ \frac{1}{2} \min |\lambda_i - \lambda_j|, \quad 1 \leq i < j \leq m, & \text{if } m > 1. \end{cases}$$

The following is an immediate corollary of Theorem 3.2:

Corollary 3.3. *Let $p(z)$ be a polynomial of degree n, $n \geq 1$, as in* (3.1)*, with roots $\{\lambda_1, \ldots, \lambda_n\}$. Then, for any ε with $0 < \varepsilon < R_0(p)$, there exists $\delta = \delta(\varepsilon) > 0$ such that the roots $\{\mu_1, \ldots, \mu_n\}$ of any degree n polynomial,*

$$q(z) = z^n + \sum_{k=0}^{n-1} b_k z^k,$$

with $|a_k - b_k| < \delta$, $k = 0, \ldots, n-1$, can be ordered such that the following holds,

$$|\lambda_i - \mu_i| < \varepsilon, \quad i = 1, \ldots, n.$$

Remark 3.4. *Let A be an $n \times n$ complex matrix. Recall that eigenvalues of A are roots of its characteristic polynomial,*

$$p(\lambda) = \det(A - \lambda I).$$

Now the coefficients of the characteristic polynomial p depend continuously on the entries of A. Therefore, it follows by Theorem 3.2 (or Corollary 3.3) that the eigenvalues of A depend continuously to the entries of A. In another words, if $\{A_n\}$ is a sequence of matrices such that $A_n \to A$, then for any $\varepsilon > 0$, there exist $N = N(\varepsilon)$ such that for $n \geq N$, the eigenvalues of A_n lie in balls of radius ε centered at the eigenvalues of A. Moreover, in the case $\varepsilon < R_0(p)$, i.e.

$$\varepsilon < \frac{1}{2} \min |\mu_i - \mu_j|, \qquad 1 \leq i < j \leq m$$

where μ_j, $j = 1, \ldots, m$ are the distinct eigenvalues of A, each having multiplicity α_j, we have the more precise result that there exists an $N = N(\varepsilon)$ such that for $n \geq N(\varepsilon)$, A_n has α_j eigenvalues in $B(\mu_j, \varepsilon)$, $j = 1, \ldots, m$.

3.3 Real roots of polynomials with real coefficients

Here we have a closer look at the real roots of a polynomial with real coefficients. The following result shows that if a polynomial p (with real coefficients) has a real root λ of multiplicity one, then any polynomial q obtained by small (real) perturbations to the coefficients of p will also have a *real* root in a neighborhood of λ. That is, not only the root λ depends continuously on coefficients of p, but also it remains real, under sufficiently small perturbations of coefficients of p. This elementary result, which is useful in applications, is not, to our knowledge, mentioned in the literature.

Theorem 3.5. *Let $p(z)$ be a polynomial of degree n, $n \geq 1$, as in* (3.1) *with* real *coefficients. Suppose λ is a* real *root of p with* multiplicity one*. Then, for any ε with $0 < \varepsilon < R_0(p)$, there exists $\delta = \delta(\varepsilon) > 0$ such that any degree n polynomial,*

$$q(z) = z^n + \sum_{k=0}^{n-1} b_k z^k,$$

with $|a_k - b_k| < \delta$, $k = 0, \ldots, n-1$, and real coefficients b_k has a real root μ of multiplicity one with $|\lambda - \mu| < \varepsilon$.

Proof. By Theorem 3.2, we know that there exits a $\delta = \delta(\varepsilon) > 0$ such that any polynomial q with $|a_k - b_k| < \delta$, $k = 0, \ldots, n-1$, has exactly one root μ in $B(\lambda, \varepsilon)$. In particular, this holds for the case that b_k are real with $b_k \in (a_k - \delta, a_k + \delta)$, $k = 0, \ldots, n-1$. Assume q is such a polynomial (with real coefficients), and suppose to the contrary that μ has non-zero imaginary part, $\mu = x + iy$, with $y \neq 0$. First we note that $\bar{\mu} = x - iy$ must also be a root of p, because $q(\bar{\mu}) = \overline{q(\mu)} = 0$, where the first equality holds since q has real coefficients. Moreover, it is simple to note that $\bar{\mu}$ is also in $B(\lambda, \varepsilon)$, as

$$|\lambda - \bar{\mu}| = \sqrt{(\lambda - x)^2 + (-y)^2} = \sqrt{(\lambda - x)^2 + y^2} = |\lambda - \mu| < \varepsilon.$$

We have thus reached a contradiction as q must have exactly one root in $B(\lambda, \varepsilon)$. Therefore, it follows that $\mu \in \mathbb{R}$ also. □

Remark 3.6. *The implication of the above result regarding the eigenvalues of a matrix is of importance also. Let $A = A(t)$ be a continuous function from $\mathbb{R}$ to the set of $n \times n$ matrices with real entries, $A : \mathbb{R} \to \mathbb{R}^{n \times n}$. Suppose $A(0)$ has a simple eigenvalue (an eigenvalue of multiplicity one) which is real; let us denote this eigenvalue by $\lambda(0)$. Then, the above Theorem says that for t sufficiently small, $A(t)$ has a* real simple eigenvalue *$\lambda(t)$, in a neighborhood of $\lambda(0)$, and as $t \to 0$, $\lambda(t) \to \lambda(0)$.*

References

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